

$$\begin{aligned}
 M_{xy} &= \iint_S \delta z \, d\sigma = \int_0^{2\pi} \int_1^2 \frac{1}{r^2} r \sqrt{2} r \, dr \, d\theta \\
 &= \sqrt{2} \int_0^{2\pi} \int_1^2 dr \, d\theta \\
 &= \sqrt{2} \int_0^{2\pi} d\theta = 2\pi\sqrt{2}, \\
 \bar{z} &= \frac{M_{xy}}{M} = \frac{2\pi\sqrt{2}}{2\pi\sqrt{2} \ln 2} = \frac{1}{\ln 2}.
 \end{aligned}$$

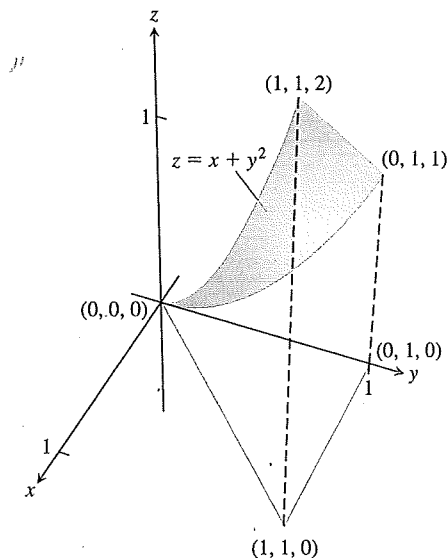
The shell's center of mass is the point $(0, 0, 1/\ln 2)$.

Exercises 16.6

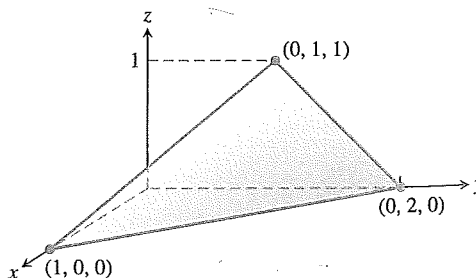
Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

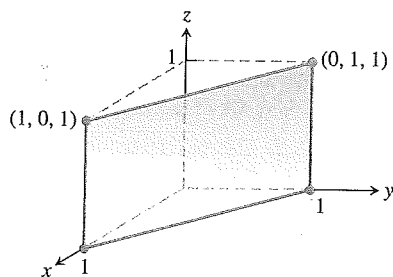
- Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2, 0 \leq x \leq 2, 0 \leq z \leq 3$
- Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4, z \geq 0, 1 \leq x \leq 4$
- Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
- Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$
- Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \leq x \leq 1, 0 \leq y \leq 1$, in the xy -plane
- Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}, 0 \leq z \leq 1$
- Parabolic dome** $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2, z \geq 0$
- Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$
- Integrate $G(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = a, y = a, z = a$.
- Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.
- Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a, y = b$, and $z = c$.
- Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a, y = \pm b$, and $z = \pm c$.
- Integrate $G(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.
- Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0, x = 1$, and $z = 0$.
- Integrate $G(x, y, z) = z - x$ over the portion of the graph of $z = x + y^2$ above the triangle in the xy -plane having vertices $(0, 0, 0), (1, 1, 0)$, and $(0, 1, 0)$. (See accompanying figure.)



- Integrate $G(x, y, z) = x$ over the surface given by $z = x^2 + y$ for $0 \leq x \leq 1, -1 \leq y \leq 1$.
- Integrate $G(x, y, z) = xyz$ over the triangular surface with vertices $(1, 0, 0), (0, 2, 0)$, and $(0, 1, 1)$.



- Integrate $G(x, y, z) = x - y - z$ over the portion of the plane $x + y = 1$ in the first octant between $z = 0$ and $z = 1$ (see the accompanying figure on the next page).



Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma$ across the surface in the specified direction.

19. **Parabolic cylinder** $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x -axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$
20. **Parabolic cylinder** $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz -plane) through the surface cut from the parabolic cylinder $y = x^2$, $-1 \leq x \leq 1$, by the planes $z = 0$ and $z = 2$
21. **Sphere** $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin
22. **Sphere** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin
23. **Plane** $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \leq x \leq a$, $0 \leq y \leq a$, in the xy -plane
24. **Cylinder** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$
25. **Cone** $\mathbf{F} = xy\mathbf{i} - z\mathbf{k}$ outward (normal away from the z -axis) through the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$
26. **Cone** $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ outward (normal away from the z -axis) through the cone $z = 2\sqrt{x^2 + y^2}$, $0 \leq z \leq 2$
27. **Cone frustum** $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ outward (normal away from the z -axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$
28. **Paraboloid** $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (normal away from the z -axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$

In Exercises 29 and 30, find the surface integral of the field \mathbf{F} over the portion of the given surface in the specified direction.

29. $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
 S : rectangular surface $z = 0$, $0 \leq x \leq 2$, $0 \leq y \leq 3$,
 direction \mathbf{k}
30. $\mathbf{F}(x, y, z) = yx^2\mathbf{i} - 2\mathbf{j} + xz\mathbf{k}$
 S : rectangular surface $y = 0$, $-1 \leq x \leq 2$, $2 \leq z \leq 7$,
 direction $-\mathbf{j}$

In Exercises 31–36, use Equation (7) to find the surface integral of the field \mathbf{F} over the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

1. $\mathbf{F}(x, y, z) = z\mathbf{k}$
2. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

33. $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$

34. $\mathbf{F}(x, y, z) = zx\mathbf{i} + zy\mathbf{j} + z^2\mathbf{k}$

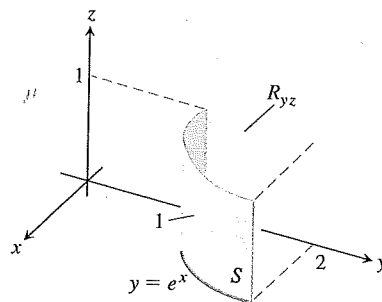
35. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

36. $\mathbf{F}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

37. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

38. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z -axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

39. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x -axis onto the rectangle R_{yz} : $1 \leq y \leq 2$, $0 \leq z \leq 1$ in the yz -plane (see the accompanying figure). Let \mathbf{n} be the unit vector normal to S that points away from the yz -plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across S in the direction of \mathbf{n} .



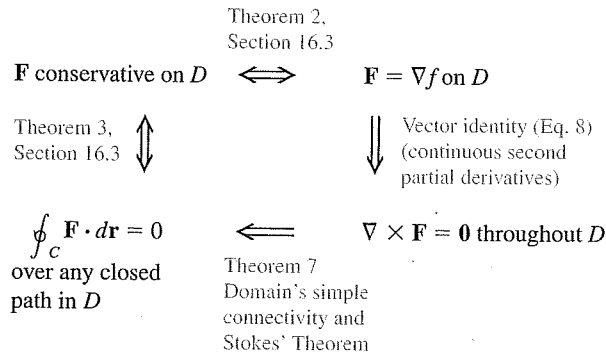
40. Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y -axis onto the xz -plane is the rectangle R_{xz} : $1 \leq x \leq e$, $0 \leq z \leq 1$. Let \mathbf{n} be the unit vector normal to S that points away from the xz -plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through S in the direction of \mathbf{n} .
41. Find the outward flux of the field $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = a$, $y = a$, $z = a$.
42. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \leq 25$ by the plane $z = 3$.

Moments and Masses

43. **Centroid** Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.
44. **Centroid** Find the centroid of the surface cut from the cylinder $y^2 + z^2 = 9$, $z \geq 0$, by the planes $x = 0$ and $x = 3$ (resembles the surface in Example 6).
45. **Thin shell of constant density** Find the center of mass and the moment of inertia about the z -axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.
46. **Conical surface of constant density** Find the moment of inertia about the z -axis of a thin shell of constant density δ cut from the cone $4x^2 + 4y^2 - z^2 = 0$, $z \geq 0$, by the circular cylinder $x^2 + y^2 = 2x$ (see the accompanying figure).

The second step is for curves that cross themselves, like the one in Figure 16.68b. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes' Theorem one loop at a time, and add the results. ■

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.



Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field \mathbf{F} around the curve C in the indicated direction.

$$\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$$

C: The ellipse $4x^2 + y^2 = 4$ in the xy -plane, counterclockwise when viewed from above

$$\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$$

C: The circle $x^2 + y^2 = 9$ in the xy -plane, counterclockwise when viewed from above

$$\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

$$\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

$$\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$$

C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy -plane, counterclockwise when viewed from above

$$\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$$

C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$, counterclockwise when viewed from above

Integral of the Curl Vector Field

Let \mathbf{n} be the outer unit normal of the elliptical shell

$$S: 4x^2 + 9y^2 + 36z^2 = 36, \quad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}} \mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

(Hint: One parametrization of the ellipse at the base of the shell is $x = 3 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$.)

8. Let \mathbf{n} be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: 4x^2 + y + z^2 = 4, \quad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1}y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, d\sigma.$$

9. Let S be the cylinder $x^2 + y^2 = a^2, 0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2, z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S .

10. Evaluate

$$\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n} \, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

Suppose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{yz}\mathbf{j} + \cos(xz)\mathbf{k}.$$

Determine the flux of \mathbf{F} outward through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

Repeat Exercise 11 for the flux of \mathbf{F} across the entire unit sphere.

as' Theorem for Parametrized Surfaces

ercises 13–18, use the surface integral in Stokes' Theorem to late the flux of the curl of the field \mathbf{F} across the surface S in the tion of the outward unit normal \mathbf{n} .

$$\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (4 - r^2)\mathbf{k}, \\ 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (9 - r^2)\mathbf{k}, \\ 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \\ 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$$

$$S: \mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + (5 - r)\mathbf{k}, \\ 0 \leq r \leq 5, \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$$

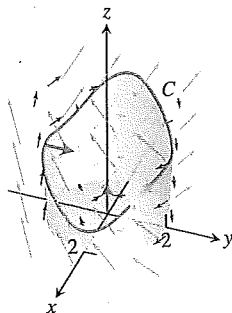
$$S: \mathbf{r}(\phi, \theta) = (\sqrt{3} \sin \phi \cos \theta)\mathbf{i} + (\sqrt{3} \sin \phi \sin \theta)\mathbf{j} + \\ (\sqrt{3} \cos \phi)\mathbf{k}, \quad 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$$

$$\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$$

$$S: \mathbf{r}(\phi, \theta) = (2 \sin \phi \cos \theta)\mathbf{i} + (2 \sin \phi \sin \theta)\mathbf{j} + (2 \cos \phi)\mathbf{k}, \\ 0 \leq \phi \leq \pi/2, \quad 0 \leq \theta \leq 2\pi$$

ery and Examples

Let C be the smooth curve $\mathbf{r}(t) = (2 \cos t)\mathbf{i} + (2 \sin t)\mathbf{j} + (3 - 2 \cos^3 t)\mathbf{k}$, oriented to be traversed counterclockwise around the z -axis when viewed from above. Let S be the piecewise smooth cylindrical surface $x^2 + y^2 = 4$, below the curve for $z \geq 0$, together with the base disk in the xy -plane. Note that C lies on the cylinder S and above the xy -plane (see the accompanying figure). Verify Equation (4) in Stokes' Theorem for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.



Verify Stokes' Theorem for the vector field $\mathbf{F} = 2xy\mathbf{i} + x\mathbf{j} + (y + z)\mathbf{k}$ and surface $z = 4 - x^2 - y^2, z \geq 0$, oriented with unit normal \mathbf{n} pointing upward.

21. **Zero circulation** Use Equation (8) and Stokes' Theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

a. $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ b. $\mathbf{F} = \nabla(xy^2z^3)$

c. $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ d. $\mathbf{F} = \nabla f$

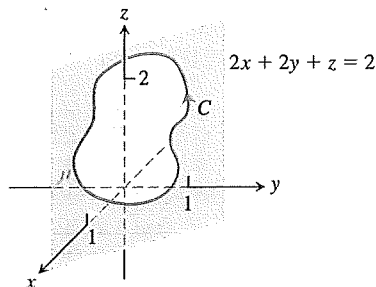
22. **Zero circulation** Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy -plane is zero

a. by taking $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \leq t \leq 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle.

b. by applying Stokes' Theorem.

23. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\oint_C 2y \, dx + 3z \, dy - x \, dz$$



depends only on the area of the region enclosed by C and not on the position or shape of C .

24. Show that if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \times \mathbf{F} = \mathbf{0}$.

25. Find a vector field with twice-differentiable components whose curl is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or prove that no such field exists.

26. Does Stokes' Theorem say anything special about circulation in a field whose curl is zero? Give reasons for your answer.

27. Let R be a region in the xy -plane that is bounded by a piecewise smooth simple closed curve C and suppose that the moments of inertia of R about the x - and y -axes are known to be I_x and I_y . Evaluate the integral

$$\oint_C \nabla(r^4) \cdot \mathbf{n} \, ds,$$

where $r = \sqrt{x^2 + y^2}$, in terms of I_x and I_y .

28. **Zero curl, yet the field is not conservative** Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy -plane. (Theorem 7 does not apply here because the domain of \mathbf{F} is not simply connected. The field \mathbf{F} is not defined along the z -axis so there is no way to contract C to a point without leaving the domain of \mathbf{F} .)